Exercise 9 – 16/12/2024

Time-dependent behavior of geomaterials – Visco-elasto-plastic stress-strain constitutive models

Problem 1 – Maxwell's model

1. Derive the differential equation, in terms of stress and strain, governing a creep or relaxation problem according to Maxwell's model, analytically.

2. A stress σ_0 is applied instantly $t_0=0$ and kept constant over time on a clay, use a Maxwell model to describe a creep process. Answer the following questions:
   a) Based on the equation obtained in question 1, provide the relationship describing the evolution of strain over time.
   b) Consider the loading condition reported in Figure 1 (total stress equal to 0.1 MPa is applied instantly at t=0, it is kept constant and is removed instantly at t=72 h) and the following values of parameters: Young's modulus: 15 GPa, viscosity coefficient: 10^4 MPa · h.

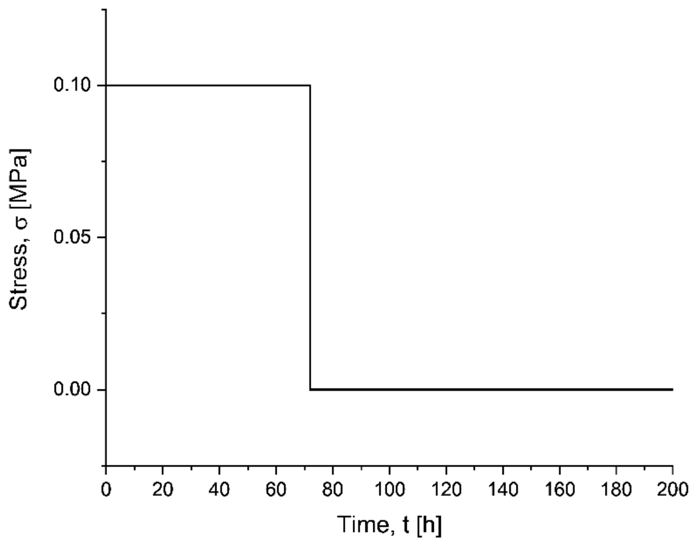

Figure 1. Loading condition.

Determine:
b1) the instantaneous strain at t=0 h;
b2) the total strain at t=24, 48 and 72, 96 h;
b3) Comment on the evolution of the total strain over time both before and after 72 h.

3. In case the stress is not removed at t=72 h, what would be the strain predicted by the model when $t \rightarrow \infty$? Is this realistic?

Solution:

1. The Maxwell model includes a spring and a dashpot in series. Since the two elements are in series, the stress in the two elements must be the same. The total strain is instead provided by the sum of the strain of the two elements.

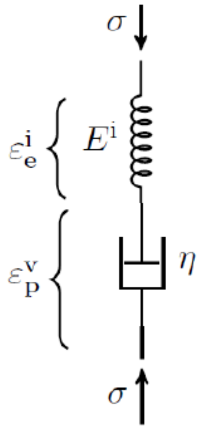

Figure I. Maxwell's model.

The equations governing the problem can therefore be written as follows (s and d subscripts stand for spring and dashpot, respectively):

$$\sigma = \sigma_s = \sigma_d \qquad (1)$$

$$\varepsilon = \varepsilon_e^i + \varepsilon_p^v \qquad (2)$$

The stress-strain constitutive relations of each element (spring and dashpot) can be written as follows:

$$\sigma_s = E^i \varepsilon_e^i \qquad (3)$$

$$\sigma_d = \eta \frac{d\varepsilon_p^v}{dt} \qquad (4)$$

The relationship between the strain of each element written in rate form reads:

$$\dot{\varepsilon} = \dot{\varepsilon}_e^i + \dot{\varepsilon}_p^v \qquad (5)$$

with

$$\dot{\varepsilon}_e^i = \frac{\dot{\sigma}_s}{E^i} \qquad \text{and} \qquad \dot{\varepsilon}_p^v = \frac{\sigma_d}{\eta}$$

It follows that it is possible writing:

$$\dot{\varepsilon} = \frac{\dot{\sigma}_s}{E^i} + \frac{\sigma_d}{\eta} \qquad (6)$$

i.e.,

$$E^i \dot{\varepsilon} = \dot{\sigma}_s + \frac{\sigma_d}{t_r} \qquad (7)$$

with $t_r = \frac{\eta}{E^i}$.

Considering that $\sigma = \sigma_s = \sigma_d$, the latter equation reads

$$E^i \dot{\varepsilon} = \dot{\sigma} + \frac{\sigma}{t_r} \tag{8}$$

or:

$$E^i \frac{d\varepsilon}{dt} = \frac{d\sigma}{dt} + \frac{\sigma}{t_r} \tag{9}$$

2.

a. In the case of an instantaneously and constant applied stress equal to σ_0, Equation (9) becomes:

$$E^i \frac{d\varepsilon}{dt} = \frac{\sigma_0}{t_r} \tag{10}$$

By integrating Equation (10) with the initial condition $\varepsilon(t_0) = \frac{\sigma_0}{E^i}$, the following solution is obtained

$$\varepsilon(t) = \varepsilon(t_0) + \frac{\sigma_0}{t_r E^i} t \tag{11}$$

i.e.,

$$\varepsilon(t) = \varepsilon(t_0) + \frac{\sigma_0}{\eta} t \tag{12}$$

b. Equation (12) allows determining the total strain at any time *t* when the constant $\sigma=\sigma_0$ is applied. The results obtained based on the provided loading condition and parameters are given in the table. Moreover, the graph showing the evolution of the strain over time, for the range 0-200 h, are also reported.

Table I. Evolution of strain over time according to Maxwell's model (Young's modulus: 15 GPa, viscosity coefficient: 10^4 MPa · h).

Time, t [h]	**Strain, ε [-]**
0	6.667E-06
24	2.467E-04
48	4.867E-04
72	7.267E-04, 7.200E-04
96	7.200E-04

At 72 h, the stress is reduced to zero. As a consequence of this, the elastic strain will be recovered (the spring reacts immediately);. In contrast, the irreversible viscous strain accumulated up to the instant t=72 h will not (i.e., the dashpot will not recover its strain). Since there is no applied stress, no further strain evolves (see Equation (4)). The strain at an instant $t > 72$ h will be equal to:

$$\varepsilon(t) = \varepsilon\left(t_{72h}\right) - \frac{\sigma_0}{E^i}. \tag{13}$$

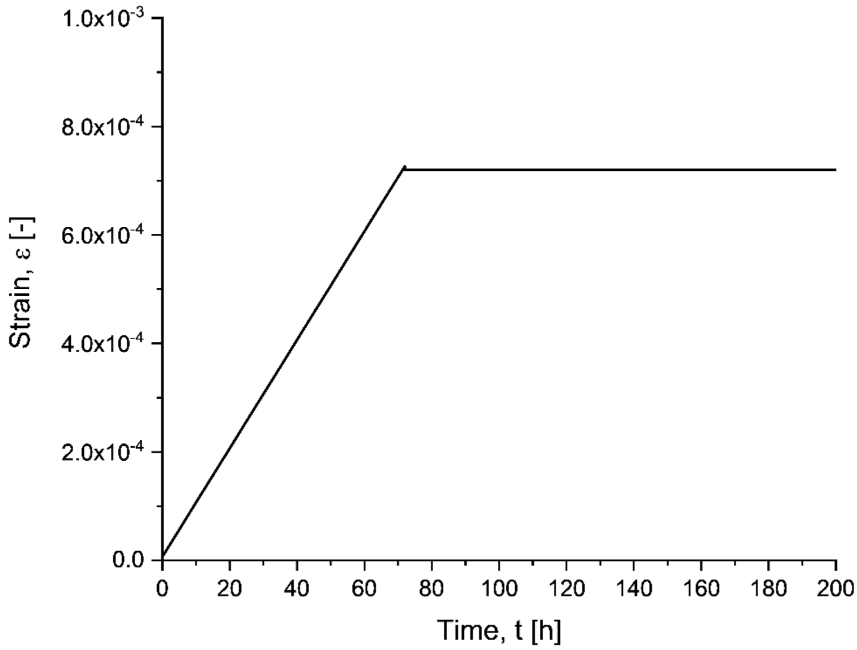

Figure II. Evolution of strain over time according to Maxwell's model.

For a creep process, Maxwell's model predicts a linear strain evolution over time (i.e., constant strain rate). This evolution is observable in the case under examination up to the instant t=72 h. When the applied stress is zero, i.e., after 72 h in the case under examination, the strain is constant over time.

3. Maxwell's model would predict a strain tending to infinity. Reaching a final finite strain value would be more appropriate.

Problem 2 – Kelvin-Voigt model

4. Derive the differential equation, in terms of stress and strain, governing a creep or relaxation problem according to the Kelvin-Voigt model, analytically.

5. A stress σ_0 is applied instantly t_0=0, and kept constant over time on granite, use a Kelvin-Voigt model to describe a creep process. Answer the following questions:
   a. Based on the equation obtained in question 4, provide the relationship describing the evolution of strain over time.
   b. Consider the loading condition reported in Figure 1 and the following parameters: Young's modulus: 20 GPa, viscosity coefficient: 10^{19} Pa · s.

      Determine:
      b1) the instantaneous strain at t=0 h;
      b2) the total strain at t=24, 48 and 72, 96 h;
      b3) Comment on the evolution of the total strain over time both before and after 72 h.

   c. Repeat the computations of question 5b for the clay considered in Problem 1 (Young's modulus: 15 GPa, viscosity coefficient: 10^4 MPa · h). Make a comparison with the predictions obtained according to the Maxwell's model.

Solution:

4. The Kelvin-Voigt model includes a spring and a dashpot in parallel. Since the two elements are in parallel, the strain in the two elements must be the same. The stress is instead provided by the sum of the stress of the two elements.

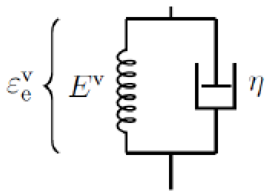

Figure III. Kelvin-Voigt model.

The equations governing the problem can therefore be written as follows (s and d subscripts stand for spring and dashpot, respectively):

$$\sigma = \sigma_s + \sigma_d \quad (13)$$

$$\varepsilon = \varepsilon_e^i = \varepsilon_e^v \quad (14)$$

The stress-strain constitutive relations of each element (spring and dashpot) can be written as follows:

$$\sigma_s = E^v \varepsilon_e^i \quad (15)$$

$$\sigma_d = \eta \frac{d\varepsilon_e^v}{dt} \quad (16)$$

Thus, the constitutive law relating stress and strain reads

$$\sigma = E^v \varepsilon_e^v + \eta \frac{d\varepsilon_e^v}{dt} \tag{17}$$

Or:

$$\sigma = E^v \varepsilon_e^v + \eta \dot{\varepsilon}_e^v \tag{18}$$

5. a. For a creep process with $\sigma=\sigma_0$, the solution of the first-order non-homogeneous differential equation (18) is

$$\varepsilon_e^v(t) = \frac{\sigma_0}{E^v} + e^{-\frac{E^v t}{\eta}} k \tag{19}$$

being k a constant.
In this case, the immediate shortening of the spring is prevented by the presence of the dashpot. Thus, the initial condition is: $\varepsilon_e^v(t_0) = 0$.
Considering the initial condition, Equation (19) becomes:

$$\varepsilon_e^v(t) = \frac{\sigma_0}{E^v}\left(1 - e^{-\frac{E^v t}{\eta}}\right) \tag{20}$$

with $t_R = \frac{E^v}{\eta}$ called, according to this model, retardation time.

b. Equation (20) allows computing the strain at any time when the constant $\sigma=\sigma_0$ is applied. The results obtained based on the provided loading condition and parameters are given in the table. Moreover, the graph showing the evolution of the strain over time, for the range 0-200 h, are also reported.

At 72 h, the stress is reduced to zero. Equation (18) becomes:

$$0 = E^v \varepsilon_e^v + \eta \dot{\varepsilon}_e^v \tag{21}$$

The solution of Equation (21) is:

$$\varepsilon_e^v(t) = e^{-\frac{E^v t}{\eta}} k \tag{22}$$

The time t is now from the time at which the unloading process begins; the strain at the beginning of the unloading process (i.e., at t= 72 h) is the one provided by equation (20):

$$\varepsilon_e^v(t_{72h}) = \frac{\sigma_0}{E^v}\left(1 - e^{-\frac{E^v t_{72h}}{\eta}}\right) \tag{23}$$

Adopting Equation (23) as the initial condition, Equation (22) becomes:

$$\varepsilon_e^v(t) = \frac{\sigma_0}{E^v} e^{-\frac{E^v t}{\eta}} \left(e^{\frac{E^v t_{72h}}{\eta}} - 1 \right) \tag{24}$$

Equation (24) can be applied for computing the strain after unloading.

Table II. Evolution of strain over time according to the Kelvin-Voigt model (Young's modulus: 20 GPa, viscosity coefficient: 10^{19} Pa · s).

Time, t [h]	**Strain, ε [-]**
0	0
24	8.639E-10
48	1.728E-09
72	2.591E-09
96	2.591E-09
200	2.589E-09

The strain evolution up to the instant t=72 h is governed by Equation (20). The rate with which the limit value $\frac{\sigma_0}{E^v}$ would be reached is dependent on $t_R = \frac{E^v}{\eta}$.

After the instant t=72 h, the equation governing the strain is Equation (24). The strain tends to zero when $t \to \infty$.

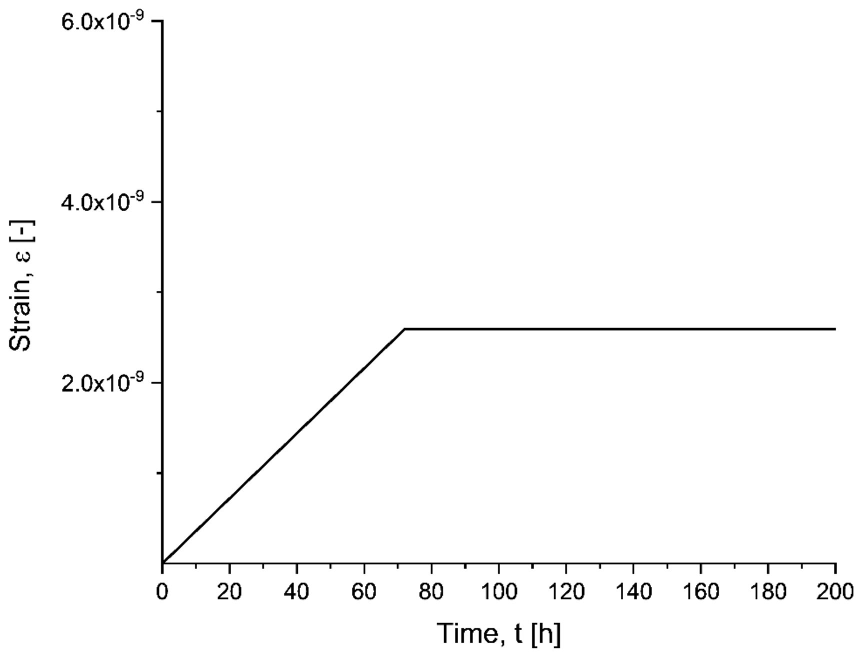

Figure III. Evolution of strain over time according to the Kelvin-Voigt model (Young's modulus: 20 GPa, viscosity coefficient: 10^{19} Pa · s).

d. The following are the results obtained according to the Kelvin-Voigt model if the following parameters are adopted: Young's modulus: 15 GPa, viscosity coefficient: 10^4 MPa · h (i.e., the same as in Problem 1).

Table III. Evolution of strain over time according to the Kelvin-Voigt model (Young's modulus: 15 GPa, viscosity coefficient: 10^4 MPa · h).

Time, t [h]	**Strain, ε [-]**
0	0
24	6.667E-06
48	6.667E-06
72	6.667E-06
96	0
200	0

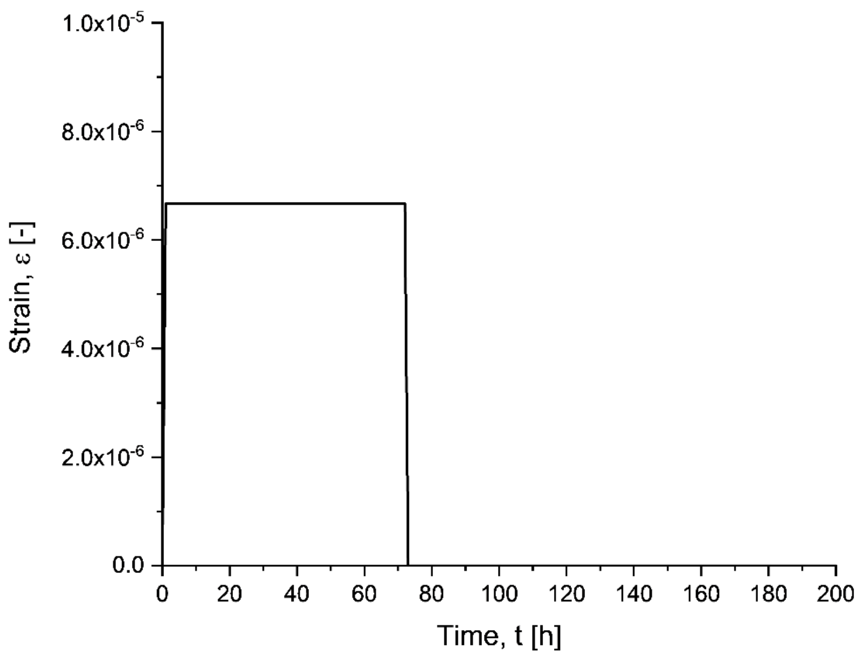

Figure IV. Evolution of strain over time according to the Kelvin-Voigt model (Young's modulus: 15 GPa, viscosity coefficient: 10^4 Pa · s).

Figure V shows the comparison between Maxwell's model and Kelvin-Voigt model, both adopted for the same geomaterial and loading condition.

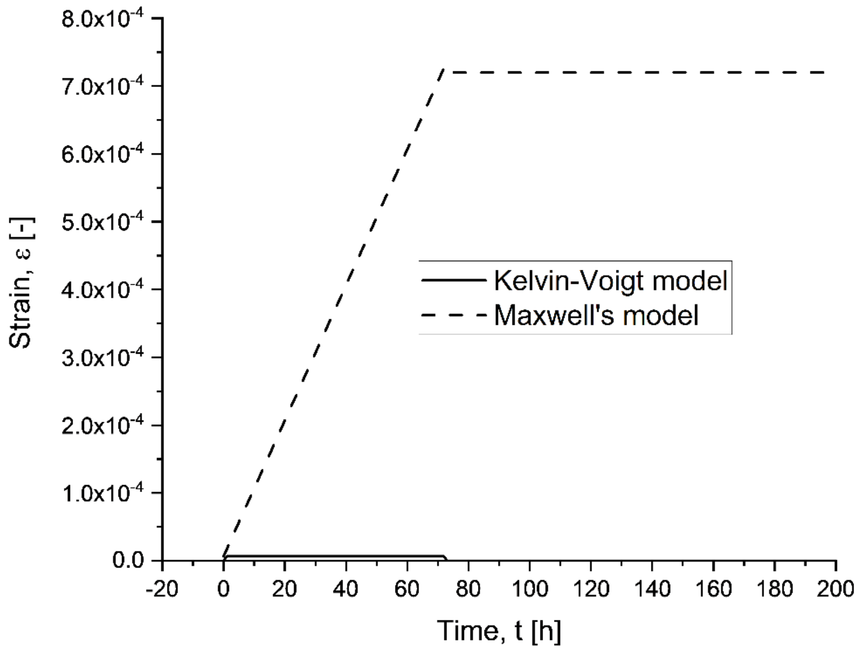

Figure V. Evolution of strain over time according to the Kelvin-Voigt model (Young's modulus: 15 GPa, viscosity coefficient: 10^4 MPa · h).

The two models foresee different strains at the same time of observation. Maxwell's model is generally not considered suitable for modeling a creep process as it would predict infinite strain after a certain interval of time. The Kelvin-Voigt model predicts the achievement of a constant strain value after a significantly high time. For the considered clay, due to the high ratio E^v / η, the strain limit value (σ_0 / E^v) is already reached from the first observation hour. Maxwell's model, unlike the Kelvin-Voigt one, predicts plastic strain.